3. Kheisin, D. E., Dynamics of Ice Sheets. Gidrometeoizdat, Leningrad, 1967.
4. Landau, L. D. and Lifshits, E. M., Theoretical Physics. Vol. 7, Theory of Elasticity. (English translation), Pergamon Press, Book № 13128, 1959.
5. Kochin, N. E., Vector Calculus and Principles of Tensor Calculus. USSR Academy of Sciences Press, Moscow, 1961.
6. Konenkov, Iu. K., Naumkina, N. I. and Tartakovskii, B. D., Investigation of forced bending vibrations of an elastic strip. Akust. Zh., Vol. 11, № 3, 1965.
7. Fedoriuk, M. V., Relationship of orthogonality type in solid waveguides, Akust. Zh., Vol. 20, № 2, 1974.

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EXISTENCE OF A GLOBAL LIAPUNOV FUNCTIONAL FOR CERTAIN CLASSES OF NONLINEAR DISTRIBUTED SYSTEMS

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The existence of a global Liapunov functional for nonlinear evolutionary equations in a Hilbert space is investigated as a continuation of paper [1]. The results obtained represent a generalization of the results of the theory of absolute stability [2, 3], for the systems with infinite dimensional phase space, and are used for investigation of the nonlocal stability and instability of nonlinear distributed systems. The conditions of existence of the global Liapunov functional obtained are illustrated by an example of a nonlinear parabolic system defined in the interval [0, 1]. The concept of a Liapunov functional was first introduced and used with success in [4].

1. Evolutionary equations in a Hilbert space. Class N of nonlinear operators. Let H. V and U be the Hilbert spaces [5] over a field of real numbers with scalar products $\langle , \rangle_H$, $\langle , \rangle_V$ and $\langle , \rangle_U$, and zero elements θ_H, θ_V and θ_U, respectively. We denote by H^* and V^* the Hilbert spaces conjugate to H and V [5], assume that $V \subset H = H^* \subset V^*$, that the space V is dense in H and, that the imbedding $V \to H$ is continuous. Let A be a continuous nonlinear operator $V \to V^*$ closed in the space H. Further, let B be a linear bounded operator $U \to V^*$ and $\Phi(\cdot)$ a nonlinear (generally speaking) operator $H \times R^1 \to U$, where R^1 denotes the real axis.

We consider the following nonlinear evolutionary equation [6]:

$$\frac{d}{dt} x(t) = Ax(t) + B\Phi(x(t), t) \tag{1.1}$$

By the generalized solution of (1.1) in the interval (τ, T) we understand the function $x(t) \in W(\tau, T; V)$ satisfying the equation

$$\int_\tau^T \left[\left\langle x(t), \frac{d\xi(t)}{dt}\right\rangle + \langle Ax(t), \xi(t)\rangle + \langle B\Phi(x(t), t), \xi(t)\rangle\right] dt = 0 \tag{1.2}$$

for any function $(\tau, T) \to V$ smooth in t and finite in the interval (τ, T). Here $W(\tau, T; V)$ is a Hilbert space of mappings $y(t) : (\tau, T) \to V$ such that $y(t) \in L^2(\tau, T; V)$, $y^{\cdot}(t) \in L^2(\tau, T; V^*)$, where $y^{\cdot}$ denotes a generalized derivative and $\langle f, g\rangle$, $f \in V^*$, $g \in V$ denotes the value of the functional f on the element g.

Definition 1. Let

$$p : H \to H,\ q : U \to H,\ r : U \to U,\ \alpha : U \to V$$

be linear bounded operators and p and r be symmetric operators

$$\langle px, y\rangle_H = \langle x, py\rangle_H,\ \langle ru, v\rangle_U = \langle u, rv\rangle_U \quad (x, y \in H,\ u, v \in U)$$

We shall say that the operator $\Phi : H \times R^1 \to U$ belongs to the class $N = N(p, q, r, \alpha)$ if the following conditions hold for this operator:

$$\langle px, x\rangle_H + 2\langle x, q\Phi(x, t)\rangle_H + \langle r\Phi(x, t), \Phi(x, t)\rangle_U \leqslant 0,\ x \in H, t > 0 \quad (1.3)$$

and a functional $W(x)$ continuous in the space H exists such that for every function $x(t) \in W(0, T; V)$ and for any interval (t_1, t_2), $t_1 < t_2$ the following inequality holds:

$$W(x(t))\Big|_{t=t_1}^{t=t_2} \geqslant \int_{t_1}^{t_2} \left\langle \alpha\Phi(x(t), t), \frac{dx}{dt}\right\rangle dt \quad (1.4)$$

Note 1. The author of paper [7] named the conditions of the type (1.3) in the case of finite-dimensional spaces H and U (with the sign reversed), the local relation connecting $x(t)$ with $\Phi(t) = \Phi(x, (t), t)$. The condition (1.4) is in fact equivalent to the condition which was called, in the above paper, the differential relation.

Assumption 1. A unique general solution $x(t) = x(t, t_0, x_0, \Phi)$, $t \in [t_0, T]$ of (1.1) satisfying the condition $x(t_0) = x_0$ exists for any $x_0 \in H$, any t_0, $T > t_0$ and $\Phi \in N$

2. Existence of the Liapunov functional for a nonlinear evolutionary equation, for the class N of nonlinear operators. Here as in [1], the essential part is played by the integral inequality

$$\int_0^\infty [\langle Ru(t), u(t)\rangle_U + 2\langle y(u)(t), Qu(t)\rangle_H + \langle Py(u)(t), y(u)(t)\rangle_H - \quad (2.1)$$
$$\varepsilon\| u(t) - Fy(u)(t)\|^2]\, dt > 0, \quad u(t) \in L^2(0, \infty; U), \quad \varepsilon > 0$$

where $y(u)(t) = y(t, \theta, u)$ is a solution of (1.1) for $\Phi_{(t)=u(t)}$, and F is an arbitrary operator such that $A + BF$ is an L^2-stable operator [1].

In [1] we have formulated a number of assumptions (assumptions 1 — 3) relating to the operators A and B. When these assumptions and the integral inequality all hold (for the L^2-stable operator A the operator F can be assumed zero), a linear continuous symmetric operator $M : H \to V$ will exist satisfying the equation

$$\langle (M^*A + A^*M + P)\xi, \eta\rangle = \langle LL^*\xi, \eta\rangle,\ L = (M^*B + Q)K^{-1},\ K^*K = \quad (2.2)$$
$$R,\ \forall\xi, \eta \in V$$

Theorem 1. Let the assumptions 1 — 3 of [1] hold for the operators A and B Let also the assumption 1 of the present paper hold and the function $\Phi(x, t)$ satisfy the condition (1.4). Let finally the integral inequality (2.1) with the operators

$$R = r + B^*\alpha,\ P = p,\ Q = q + {}^1/_2 A^*\alpha \quad (2.3)$$

hold for the linear bounded operators p, q and r , $H \to H$, $U \to H$, $U \to U$. Here $B^*\alpha$ and $A^*\alpha$ are bounded operators and the operator F is such that the operator $A + BF$ is L^2-stable [1]. Then the continuous functional

$$V(x) = \langle Mx, x\rangle_H + W(x) \tag{2.4}$$

exists in the space H such that for any solution $x(t) = x(t, t_0, x_0, \Phi)$ of (1.1) and for any value of Φ , the inequality

$$V(x(t))\Big|_{t=t_1}^{t=t_2} \geqslant \int_{t_1}^{t_2} (\Sigma(x(t))\,dt, \quad \Sigma(x(t)) \equiv \| L^*x(t) + K\Phi(t)\|_U^2 - \tag{2.5}$$
$$[\langle px(t), x(t)\rangle_H + 2\langle x(t), q\Phi(t)\rangle_H + r\langle \Phi(t), \Phi(t)\rangle_U]$$

holds.

If for the same $\Phi \in N(p_\varepsilon, q, r_\delta, \alpha) = N_{\varepsilon,\delta}$, $p_\varepsilon = p + \varepsilon E_H$, $r_\delta = r + \delta E_U$ (E_H and E_U are identity operators in H and U), then for $V(x)$ we have the following inequality:

$$V(x(t_2)) - V(x(t_1)) \geqslant \int_{t_1}^{t_2} (\varepsilon \|x(t)\|_H{}^2 + \delta \|\Phi(t)\|_U{}^2)\,dt, \quad t_1 < t_2 \tag{2.6}$$

Proof. The inequality (2.5) is obtained from the following sequence of relations which are true by virtue of (1.1), (1.4) and (2.2) — (2.4):

$$\int_{t_1}^{t_2} \Sigma(x(t))\,dt = \int_{t_1}^{t_2} [\langle M^*Ax(t), x(t)\rangle + \langle M^*B\Phi(t), x(t)\rangle + \langle Mx(t), Ax(t)\rangle +$$
$$\langle Mx(t), B\Phi(t)\rangle + \langle Ax(t), \alpha\Phi(t)\rangle + \langle B\Phi(t), \alpha\Phi(t)\rangle]\,dt =$$
$$\int_{t_1}^{t_2} \left[\left\langle M^* \frac{dx(t)}{dt}, x(t)\right\rangle + \left\langle Mx(t), \frac{dx(t)}{dt}\right\rangle + \left\langle \frac{dx(t)}{dt}, \alpha\Phi(t)\right\rangle\right] dt \leqslant$$
$$\int_{t_1}^{t_2} \frac{d}{dt}\langle Mx(t), x(t)\rangle\,dt + W(x(t))\Big|_{t_1}^{t_2} = V(x(t))\Big|_{t_1}^{t_2}$$

Here and in what follows, we omit the indices accompanying the spaces in the scalar products and norms.

From the first relation of (2.2) we see that the properties of the operator M depend on the properties of the homogeneous linear equation

$$\frac{d}{dt} y = Ay \tag{2.7}$$

Let us denote by $y(t, y_0)$ the solution of (2.7) corresponding to the initial condition $y(0) = y_0$. (Here we regard the solution in the same sense as that of (1.1) for $\Phi \equiv \theta$. The existence and uniqueness of this solution at $t > 0$ is guaranteed by the assumption 1 of [1]).

Assumption 2. Let the space H be decomposable into a direct sum of subspaces H_+ and H_- defined by the following properties:

a) the solution $y_+(t) = y(t, y_0)$ of (2.5) satisfies for any $y_0 \in H_+$ the condition

$$\lim_{t\to\infty} y_+(t) = \theta \tag{2.8}$$

b) for every $y_0 \in H_-$ there exists a unique solution $y_-(t) = y(t, y_0)$ of (2.5) defined in the interval $(-\infty, 0)$ and satisfying the condition

$$\lim_{t\to-\infty} y_-(t) = \theta \tag{2.9}$$

We shall call the operator L positive, $L > 0$ (negative, $L < 0$) if $\langle Lh, h\rangle \geqslant 0$ ($\langle Lh, h\rangle \leqslant 0$). We have $\langle Lh, h\rangle = 0$ only for $h = \theta$.

Theorem 2. Let us assume that all the conditions of Theorem 1 and the Assumption 2 hold and that the operator $p - LL^*$ is of constant sign. Then the operator M will be sign-constant on the subspaces H_- and H_+. On H_+ its sign will be the same as that of the operator $p - LL^*$, and on H_- it will be the opposite.

Proof. We assume, for definiteness, that $p - LL^* > 0$. Let us consider the solution $y(t) \not\equiv \theta$ of (2.7) defined in some interval (t_1, t_2). Then, taking (2.2) into account, we have

$$\langle My(t), y(t)\rangle \Big|_{t_1}^{t_2} = \int_{t_1}^{t_2} \langle (M^*A + A^*M)\, y(t), y(t)\rangle\, dt = \tag{2.10}$$
$$\int_{t_1}^{t_2} \langle (p - LL^*)\, y(t), y(t)\rangle\, dt > 0$$

Let $y_0 \in H_+$, and consider the relation (2.10) for the solution $y(t, y_0)$, $t > 0$. Directing t_2 to infinity and t_1 to zero and paying due regard to (2.8), we obtain

$$\langle My_0, y_0\rangle = \int_0^{\infty} \langle (p - LL^*)\, y(t), y(t)\rangle\, dt > 0, \quad y_0 \neq \theta \tag{2.11}$$

Similarly, considering $y_0 \in H_-$ and $t_1 < t_2 < 0$ we obtain from (2.10) (after passing to the limit with $t_2 \to 0$ and $t_1 \to -\infty$ and taking into account (2.9)), the proof of the theorem also for the subspace H_-.

From Theorems 1 and 2 we arrive at the following obvious result.

Theorem 3. Let the conditions of Theorem 1 and 2 all hold, and $p < 0$. Then for any operator $\Phi \in N$ ($\Phi \in N_{\varepsilon,\delta}$) a functional (2.4) will exist for the equation (1.1) satisfying the inequality (2.5) (inequality (2.6)). When the functional $W(x)$ is negative, the functional (2.4) will be positive on the subspace H_-, and when the functional $W(x)$ is nonpositive, (2.4) will be negative on the subspace H_+.

3. A particular case when p is a zero operator. In this case (2.10) implies that $M \leqslant 0$ on the subspace H_+ and $M \geqslant 0$ on the subspace H_-. Let us investigate the properties of the linear spaces $L^{\pm}(M) = \{x \mid x \in H_{\pm},\ Mx = \theta\}$. In particular, we wish to find out in which case these spaces are zero-dimensional.

Assumption 3. The pair (A, Q^*) of operators has the following property on the subspace H_+: from $Q^*y(t, y_0) = \theta_U$, $y_0 \in H_+$, it follows that $y_0 = \theta_H$. We note that in the case of finite-dimensional spaces H and U the above property coincides with the property of identifiability of the pair (A, Q^*) according to Kalman.

Theorem 4. Let A be an L^2-stable operator [1]. Let also the conditions of Theorem 2 (*) all hold for $p = [0]$, as well as the Assumption 3. Then the linear space $L^+(M)$ consists of a single zero element.

Proof. Let h be any arbitrary element of the space L^+. Then the minimum value of the functional $I_h[u]$ of the variational problem of Theorem 3 in [1] will be $\langle Mh, h\rangle = 0$. Since $p = [0]$, the function $u(t) \equiv \theta$ will be the only extremal element $u^{\circ}(t)$ of this problem the existence of which is guaranteed by Theorem 3 of [1]. Moreover, the

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*) The operator F in the integral inequality can be assumed zero by virtue of the L^2-stability of the operator A.

following relations [1] will hold for the functions $y^{\circ}(t) = y(t, u^{\bullet})$ and $\Psi(t) = My^{\bullet}(t)$:

$$\frac{d}{dt}\Psi = -A^{*}\Psi, \quad \frac{dy^{\circ}}{dt} = Ay^{\circ}, \quad B^{*}\Psi + Q^{*}y^{\circ} = \theta, \quad t > 0 \tag{3.1}$$

From (3.1) and the assumption 2 of [1] (which appears in the conditions of Theorems 3 and 4), we obtain the following relations: $\Psi(t) = \theta$, $Q^{*}y^{\circ}(t) = \theta$, $y(0) = h = \theta$. The latter proves the theorem.

From Theorems 3 and 4 follows

Theorem 5. Let us assume that $H = H_{+}$, $W(x) \leqslant 0$ and that the conditions of Theorem 4 are satisfied. Then for Eq. (1.1) for any $\Phi \in N([0], q, r, \alpha)$ there exists the negative Liapunov functional (2.4) for which the inequalities (2.5) and (2.6) hold. If $W(x) \leqslant -c\|x\|^{q}$, $c > 0$, then $V(x) \leqslant -\text{const}\,\|x\|^{q}$.

4. Example. We consider, as a simple but nontrivial example, the partial differential equation in the interval [0, 1] (of a parabolic type)

$$\frac{\partial u(s,t)}{\partial t} = a(s)\frac{\partial^{2}u(s,t)}{\partial s^{2}} - b(s)u(s,t), \quad 0 < s < 1 \tag{4.1}$$

$$\left.\frac{\partial u}{\partial s}\right|_{s=0} = 0, \quad \left.\frac{\partial u}{\partial s}\right|_{s=1} = \varphi(w(t))$$

$$w(t) = G(u(s,t)) = \int_{0}^{1} g(s)u(s,t)\,ds \tag{4.2}$$

$$a(s) \geqslant a > 0, \quad b(s) \geqslant b, \quad a(s) \in C^{1}(0,1) \tag{4.3}$$

$$\mu_{1} \leqslant \frac{\varphi(w)}{w} \leqslant \mu_{2} \quad (w \neq 0), \quad \int_{0}^{x} \varphi(\xi)\,d\xi \geqslant \text{const}\,\|x\|^{q} \tag{4.4}$$

$$q > 0, \quad \mu_{1} \leqslant 0, \quad \mu_{2} > 0$$

Here G is a linear continuous functional acting in the space $L^{2}(0,1)$ of square summable functions defined in the interval (0, 1), and $\varphi(w)$ is the continuous function satisfying the conditions given above. We shall consider solutions of Eqs. (4.1), (4.2) generalized in the Sobolev sense. Namely, under a solution generalized in the interval $t \in [0, T]$ we understand the function $u(s, t)$ satisfying the following conditions:

$$\int_{0}^{1}\left(|u(s,t)|^{2} + \left|\frac{\partial u}{\partial s}(s,t)\right|^{2}\right)ds < \infty, \quad 0 < t < T \tag{4.5}$$

Here $\partial / \partial s$ denotes a generalized derivative, or a derivative in the sense of the theory of distribution; the following relation holds for any function $g(s, t)$ satisfying the condition (4.5), smooth and finite in t in the interval $[0, T]$:

$$\int_{0}^{T} dt\left\{\int_{0}^{1}\left[u(s,t)\frac{\partial g(s,t)}{\partial t} - \left(a(s)\frac{\partial u}{\partial s}\frac{\partial g}{\partial s} + b(s)ug\right)\right]ds + a(1)\varphi(w(t))g(1;t)\right\} = 0 \tag{4.6}$$

In order to include the above equation in the framework of the proposed theory, we must first write it in the form of (1. 1). This requires adequate definition of the spaces H, U and V, and of the operators A, B and Φ. Secondly, we must introduce the operators p, q, r and α defining the class N. As H, we shall use the space $L^2(0, 1)$ of functions $u = u(s)$ square summable in the interval (0, 1) with the scalar product

$$\langle u, v\rangle_H = \int_0^1 u(s)\, v(s)\, ds \tag{4.7}$$

(More accurately, the elements of the space $L^2(0, 1)$ are not functions, but classes of functions [5, 6]).

As V, we take the space $H^1(0, 1)$ of functions $u(s)$, $s \in (0,1)$ satisfying the inequality (4. 5), with the scalar product [6]

$$\langle u, v\rangle_V = \int_0^1 \left(u(s)\, v(s) + \frac{\partial u(s)}{\partial s} \frac{\partial v(s)}{\partial s} \right) ds \tag{4.8}$$

We defining the operator $A : V \to V^*$ as follows: we set for any functions $u(s)$, $v(s) \in H^1(0,1)$

$$\langle Au, v\rangle \equiv \int_0^1 (Au)(s)\, v(s)\, ds = -\int_0^1 \left(a(s) \frac{\partial u}{\partial s} \frac{\partial v}{\partial s} + b(s)\, uv \right) ds \tag{4.9}$$

Further, let $U = R^1$, where R^1 is a unique straight line and the operator $B : U \to V^*$ is defined in accordance with the equation

$$\langle B\xi, v(s)\rangle = a(1)\, \xi v(1) \qquad (\xi \in R^1,\ v(s) \in H^1(0, 1)) \tag{4.10}$$

(If we agree to denote the operators of multiplication by any function or the operator σ by $[\sigma]$, then according to this notation $B = [a(1)\delta(s - 1)]$, where $\delta(s)$ is the Dirac delta function).

Finally we define the operator $\Phi : L^2 \to R^1$ as the following composition of operators:

$$\Phi : L^2 \ni u(s) \to w = G(u(s)) \to \varphi(w) \in R^1 \tag{4.11}$$

When the operators A, B and Φ are defined in this manner, the solution of the evolutionary equation (1. 1) will coincide with the generalized solution of (4. 1), (4. 2) understood in the sense of (4. 5), (4. 6). Let us now define the operators p, q, r and α. By (4.4), the function $\varphi(w)$ satisfies the relations

$$(\mu_2 w - \varphi(w))(\mu_1 w - \varphi(w)) \leqslant 0, \qquad \int_{w(t_1)}^{w(t_2)} \varphi(\lambda)\, d\lambda = \int_{t_1}^{t_2} \varphi(w)\, w^{\cdot} dt \tag{4.12}$$

From (4. 2), (4. 4) and (4. 12) it follows that $\Phi \in N(p, q, r, \alpha)$, where

$$p = \mu_1\mu_2 G^* G, \quad r = [1], \ q = -{}^1/_2\, (\mu_1 + \mu_2)\, G^*, \quad \alpha = \rho G^* \tag{4.13}$$

$$W(u(s)) = \rho \int_0^w \varphi(\lambda)\, d\lambda, \quad w = G(u(s)) \tag{4.14}$$

Next we shall explain what form inequality (2. 1) assumes in the present example. We denote by $\chi(p)$ a function of the complex variable p defined by the relation

$$\chi(p) = G(u_1(s, p)), \quad a(s) \frac{\partial^2 u_1}{\partial s^2} - (b(s) + p)\, u_1 = 0 \tag{4.15}$$

$$\left.\frac{\partial u_1}{\partial s}\right|_{s=0} = 0, \quad \left.\frac{\partial u_1}{\partial s}\right|_{s=1} = 1 \quad (p = \alpha + j\omega)$$

(In the control theory this function is known as the transmission coefficient in the system (4. 1), (4. 2), from the quantity φ to the quantity w [2]). Using (4.9)—(4.11), (4.13) and (4. 15) and performing simple manipulations we can readily establish that the inequality (2. 1) is equivalent to the following inequality well known in the theory of absolute stability

$$\operatorname{Re}\left[(\mu_2\chi(j\omega)-1)(\mu_1\chi(j\omega)-1)+\rho j\omega\chi(j\omega)\right]\geqslant\varepsilon>0\qquad(\omega\in(-\infty,+\infty))\quad(4.16)$$

Let us now analyze the assumptions 1 — 3 of [1] and the assumptions 1 — 3 of the present paper which appear in Theorems 1 — 5. In accordance with (4. 3) and (4. 9), the operator A satisfies the inequality

$$\langle -Au, u\rangle\geqslant a\left\|\frac{\partial u}{\partial s}\right\|_H^2+b\|u\|_H^2\geqslant\varepsilon\|u\|_V^2+\lambda\|u\|_H^2,\quad\varepsilon>0\qquad(4.17)$$

Consequently [6] the assumption 1 of [1] on the existence and uniqueness of the solution will hold; when $b>0$ in (4. 17), we have $\lambda>0$ and the operator A will be L^2-stable.

The assumption 2 of [1] can, in view of (4. 8) and (4. 9), be formulated for the present example as follows. For any function $f(s,t)$ such that $\|f(s,t)\|_H\in L^2(0,\infty)$, there exists a unique generalized solution $\Psi(s,t)$ of the equation

$$-\frac{\partial\Psi}{\partial t}=a(s)\frac{\partial^2\Psi}{\partial s^2}-b(s)\Psi+f(s,t)=0,\quad t>0,\quad \left.\frac{\partial\Psi}{\partial s}\right|_{s=0,1}=0\qquad(4.18)$$

satisfying the conditions

$$\int_0^\infty\int_0^1\Psi^2(s,t)\,ds\,dt<\infty\qquad(4.19)$$

$$\int_0^\infty\int_0^1\frac{\partial\Psi}{\partial t}(s,t)\,g(s,t)\,ds\,dt<\infty,\quad\forall g(s,t)\in L^2(0,\infty;H^1)$$

$$\int_0^1\left(\frac{\partial\Psi}{\partial s}(s,t)\right)^2ds<\infty,\quad t>0\qquad(4.20)$$

The existence and uniqueness of the solution $\Psi(s,t)$ satisfying (4. 19) follows from the inequality (4. 17) and from Sect. 6. 2, chap. 3 of [6]. We show the validity of (4. 20) by considering the equation for $\Psi(s,t)=\partial\Psi(s,t)/\partial s$ obtained from (4. 18) by differentiating with respect to s. (We refer the function Ψ itself to the right-hand side of the equation for Ψ'.) We can show that the results of [6] mentioned above also apply to the equation obtained, and we therefore conclude that the function $\partial\Psi/\partial s$ is L^2-continuous in t and (4. 20) holds. The assumption 3 in the present example means that the solution $u(s,t)$ of the equation

$$\frac{\partial u}{\partial t}=a(s)\frac{\partial^2 u}{\partial s^2}-b(s)u,\quad t>0,\quad s\in(0,1)\qquad(4.21)$$

$$\left.\frac{\partial u}{\partial s}\right|_{s=0}=0,\quad\left.\frac{\partial u}{\partial s}\right|_{s=1}=\varphi(t),\quad\int_0^\infty\varphi^2(t)\,dt<\infty$$

will have a generalized derivative $\partial u/\partial s\in L^2(0,1)$, L^2-continuous in t, provided that the initial conditions used are fairly smooth, e. g. $u(s,0)\in H^1(0,1)$. The last assumption, as well as the previous one, can be proved by considering the equation for $\partial u/\partial s$ and using the results of [6].

Let us now pass to the assumptions made in the present paper. Assumption 1 means that a generalized solution of the boundary value problem (4. 1) — (4. 3) exists and is unique. This property is connected with the smoothness of the nonlinear function φ and the form of the operator G. Without going into a detailed investigation of this problem (about which an extensive bibliography exists, see e. g. [8]), we shall simply state that the assumption holds.

Next we check the validity of the assumptions 2 and 3 (the latter is necessary only when $\mu_1 = 0$.) These assumptions concern a homogeneous linear equation (Eq. (4. 21) with $\varphi(t) = 0$). According to [8], a solution $v(s, t)$ of this equation can be written in the form of a series

$$v(s,t) = \sum_{i=1}^{\infty} C_i e^{\lambda_i t} F_i(s) \tag{4.22}$$

where $F_n(s)$ are the eigenfunctions of the self-conjugate operator

$$S(v) = \{a(s)\, v''(s) - b(s)\, v(s), \quad s \in (0, 1), \quad v'(0) = v'(1) = 0\} \tag{4.23}$$

and λ_n denote the corresponding eigenvalues. Assumption 2 will hold if $\lambda_n \neq 0$, and H_+ is the subspace of $L^2(0, 1)$ stretched over the eigenfunctions corresponding to the negative values of λ_n, while H_- is the subspace stretched over the eigenfunctions corresponding to the positive λ_n (the number of the latter values is finite).

When $b > 0$, all eigenvalues are negative and H_- is empty. Let us consider the assumption 3 for this case. This assumption means that $G(v, (s, t)) \equiv 0,\ t > 0$ implies $v(s, t) \equiv 0$. From (4. 22) we see that this will indeed occur if

$$G(F_n(s)) \neq 0 \ (n = 1, 2, \ldots)$$

Thus from Theorems 3 and 5 we obtain

Theorem 6. Let the following statements holds for the boundary value problem (4. 1) — (4. 4.):

1) for any initial function $u_0(s) \in L^2(0, 1)$ there exists a unique generalized solution $u(s, t),\ t \in [0, \infty)$ which becomes equal to $u_0(s)$ at $t = 0$;

2) the operator G is a linear functional bounded on $L^2(0, 1)$ satisfying the condition (2. 24) and the condition that $[\rho\delta(s - 1)]\, G^*$ and $[\rho\partial^2 / \partial s^2]\, G^*$ represent the bounded mappings $R^1 \to R^1$ and $R^1 \to L^2$, respectively. We note that if $G(v) = \int g(s)\, v(s)\, ds$, $g(s)$ is twice integrable in the interval (0, 1) and $|g''(s)| < \infty,\ s \in [0, 1]$, then the latter conditions will also hold when $\rho \neq 0$;

3) the inequality (4. 16) holds;

4) the condition of stabilizability holds. A function $\rho(x) \in L^2(0, 1)$ exists such that all solutions of the linear equation (4. 1) — (4. 4) with $\varphi(u) = \int \rho(s)\, u(s, t)\, ds$ tend exponentially to zero over the norm of $L^2(0, 1)$. (If $b > 0$, then $p(s) = 0$.)

Then a continuous functional $V(u(s)) : L^2(0, 1) \to R^1$ of the form

$$V(u(s)) = \int_0^1 (Mu)(s)\, u(s)\, ds + \rho \int_0^w \varphi(\sigma)\, d\sigma$$

will exist (M is a linear bounded operator $L^2(0, 1) \to H^1(0, 1)$) which can be determined with the help of Theorem 3 of [1] and in which ρ is the quantity appearing in the condition (4. 16). The functional increases monotonously along the generalized solutions of the equations (4. 1) — (4. 4), provided that $\varphi(x)$ satisfies (4. 5) and also satisfies (2.6) if $\varphi \in N[p_\varepsilon, q, r_\delta, \alpha]$.

If $\rho \leqslant 0$ and $b > 0$, then $V(u) \leqslant -|\rho|\,\mathrm{const}\,|w|^q < 0$. The functional $V(u)$ will have positive values if $\rho \geqslant 0$ and if some of the eigenvalues λ_n of the operator (4. 23) are positive and none are zero. In the latter case we shall have the initial conditions under which the functions $V(u(s, t))$, $|\rho w(t)|$ will be unbounded functions of time.

REFERENCES

1. Brusin, V. A., Equations of Lur'e in Hilbert space and their solvability. PMM Vol. 40, № 5, 1976.
2. Gantmakher, F. R. and Iakubovich, V. A., Absolute stability of nonlinear controlled systems. Proc. of the 2nd All-Union Conference on Theoretical and Applied Mechanics. Ed. 1, Moscow, "Nauka", 1965.
3. Aizerman, M. A. and Gantmakher, F. R., Absolute Stability of Controlled Systems. Moscow, "Nauka", 1963.
4. Krasovskii, N. N., On the application of the second method of Liapunov to equations with time lag. PMM Vol. 20, № 3, 1956.
5. Yosida, K., Functional Analysis. Springer-Verlag, Berlin, Heidelberg, N. Y., 1974.
6. Lions. J. L., Optimal Control of Systems Described by Partial Differential Equations. N. Y., Berlin, Springer-Verlag, 1971.
7. Iakubovich, V. A., Absolute instability of nonlinear control systems. I. General frequency criteria. Avtomatika i telemekhanika, № 12, 1970.
8. Ladyzhenskaia, O. A., Solonikov, V. A. and Ural'tseva, N. N., Linear and Quasi-linear Equations of Parabolic Type. Moscow, "Nauka", 1967.

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PRESSURE OF A PLANE STAMP OF NEARLY CIRCULAR CROSS SECTION ON AN ELASTIC HALF-SPACE

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An approximate method of solving the contact problem of impressing a plane stamp of nearly circular cross section into an elastic half-space is suggested. The friction of the contact surface is neglected. A numerical algorithm for the method is produced. An elliptical and rectangular stamps are considered as examples.

There is no general method of solving the problems for stamps of nearly circular cross section. Apart from the classical problem of a plane elliptical stamp, the literature gives solutions for the problems of polygonal stamps, with each problem however requiring a different approach. An approximate solution for the problem of impressing a stamp of nearly circular cross section into an elastic half-space is given in [1]. The method makes it possible to use the same approach to solve the contact problem for an arbitrary region of contact, and to